ELSEVIER

Journal of Hazardous Materials 57 (1998) 169–175

JOURNAL OF HAZARDOUS MATERIALS

# Risk assessment
# A note on $F$–$n$ curves, expected numbers of fatalities, and weighted indicators of risk

I.L. Hirst

*Major Hazards Assessment Unit, Chemical and Hazardous Installations Division, Health and Safety Executive, St. Anne's House, Stanley Precinct, Bootle, Merseyside, UK*

Received 24 April 1997; received in revised form 18 June 1997; accepted 20 June 1997

**Abstract**

The paper considers one of the weighted risk indicators used by the Major Hazards Assessment Unit (MHAU) of the Health and Safety Executive in formulating advice to local planning authorities on the development of land in the vicinity of hazardous installations. It is shown that MHAU's 'Risk Integral' can be recast into the form of an expected utility function, suitable according to classical decision theory for reaching consistent decisions on risk tolerability. It is also shown that the weightings that the Risk Integral implicitly attributes to multiple fatality accidents are not out of line with those proposed by others. Crown Copyright © 1998 Published by Elsevier Science Ltd.

*Keywords:* Quantitative risk assessment; Major hazards; Societal risk; Aversion

## 1. Introduction

$F$–$n$ curves and $f$–$n$ curves are concepts in current thinking on the assessment of risks to populations from hazardous installations. It has been suggested (Evans and Verlander, Ref. [1]) that whilst $F$–$n$ curves may be valuable presentational devices, their simple use as criteria for tolerability of risk is unsatisfactory, and that a criterion based on an 'expected utility function', giving appropriate weight to multiple fatality accidents, is to be preferred. According to classical decision theory, tolerability decisions should be made on the basis of expected utility if their consistency is to be assured (e.g. Lindley, Ref. [2]).

In decision theory, 'utility' is a number expressing the merit or attractiveness of a consequence. If there are $N$ different consequences with frequencies $f(1), \ldots, f(N)$

0304-3894/98/$19.00 
*PII* S0304-3894(97)00080-0

and utilities $u(1), \ldots, u(N)$, then the expected utility function is $\Sigma f(i) \cdot u(i)$, summed from $i = 1$ to $N$. In the present context, where the consequences are accidents with multiple fatalities, the term 'disutility' seems more appropriate, and will be used from here on, following Ref. [1].

Over recent years, the Major Hazards Assessment Unit (MHAU) of the UK's Health and Safety Executive has been advising local planning authorities on land-use in the vicinity of hazardous installations, using an approach described publicly in a discussion document (Health and Safety Executive, Ref. [3]), a document which is now being revised and updated to take account of feedback and technical progress. One technical development has been MHAU's 'Risk Integral' (within Carter, Ref. [4]). The purpose of this note is to derive certain relationships which show that the Risk Integral is an expected disutility function, and that the weightings which it attributes to multiple fatality accidents are not out of line with those proposed by others.

## 2. Analysis

We begin by considering $f$–$n$ and $F$–$n$ curves. Following the usual definitions, $f(n)$ is taken to denote the frequency of accidents at a major hazard installation which cause exactly $n$ fatalities, and the $f$–$n$ curve is simply a plot of the values of $f(n)$ against $n$. Similarly, $F(n)$ is taken to denote the summed frequency of all accidents at the installation which cause $n$ or more fatalities, and the $F$–$n$ curve is a plot of the values of $F(n)$ against $n$.

A simple measure of the risk from the installation can be obtained by calculating the expected number of fatalities per year (ENFY) from accidents. It is calculated from the values of $f(n)$ as:

$$\text{ENFY} = \Sigma f(n) \cdot n$$

This has the form of an expected disutility function, with the number of fatalities $n$ being used as the measure of disutility.

The ENFY could be used in a tolerability criterion, the risk from the installation being judged tolerable if the ENFY is less than some agreed criterion value. However, a criticism that could be made of using the ENFY as a criterion is that it does not include an allowance for aversion to multiple fatality accidents. By not distinguishing between an installation which has one accident causing 100 deaths and an installation which has 100 accidents each causing one death over the same period of time, the ENFY fails to reflect the contrast between society's strong reaction to major accidents that occur occasionally and its quiet tolerance of the many small accidents that occur frequently.

As a remedy, risk assessment practitioners have suggested using instead of the ENFY, a 'weighted risk indicator' which gives greater emphasis to the number of deaths in an accident by raising the disutility $n$ in the evaluation to some power greater than one. Alternatives to the ENFY have included (Schofield, Ref. [5]):

$$\Sigma f(n) \cdot n^{1.5} \text{ and } \Sigma f(n) \cdot n^2$$

These too have the form of an expected disutility function. The factor by which each term in the ENFY summation has been enhanced can be called the 'aversion multiplier'. In the cases shown, the aversion multiplier is $n^{0.5}$ and $n$, respectively.

An earlier reference (Okrent, Ref. [6]) tentatively proposed $\Sigma f(n) \cdot n^{1.2}$.

The practice in MHAU is to use the expression

$$\Sigma F(n) \cdot n.$$

When written in integral form, this expression is called by MHAU as the 'Risk Integral'. At first sight, the expression may appear to be a radically different way of introducing aversion, but in fact, MHAU's method will be shown below to correspond closely to the other methods, certainly for those forms of $F$–$n$ curves that are of practical interest.

We begin by considering an installation which has an $F$–$n$ curve of the form usually presented. This is a straight line plotted on log–log axes and extending to very large values of $n$ without limitation.

Firstly, from the definitions of $f(n)$ and $F(n)$ we obtain:

$$f(n) = F(n) - F(n+1) \tag{1}$$

for $n = 1, 2, 3, \ldots$ and so on.

Secondly, for this form of $F$–$n$ curve, we have:

$$F(n) = \frac{F(1)}{(n)^{a}} \tag{2}$$

where $a$ is a constant (usually of the order of 1 or 2), and again $n = 1, 2, 3, \ldots$ and so on.

Using Eq. (2) to substitute for $F(n+1)$ in Eq. (1) gives:

$$f(n) = F(n) - \frac{F(1)}{(n+1)^{a}}.$$

Being imaginative with the last term we can rewrite this as:

$$f(n) = F(n) - \frac{F(1)}{(n)^{a}} * \frac{(n)^{a}}{(n+1)^{a}}.$$

Using Eq. (2) again to replace part of the last term gives:

$$f(n) = F(n) - F(n) \times \frac{(n)^{a}}{(n+1)^{a}}.$$

Finally, combining the $F(n)$ terms and isolating $F(n)$ gives:

$$F(n) = f(n) \times \left[ \frac{(n+1)^{a}}{(n+1)^{a} - (n)^{a}} \right].$$

The last result tells us how corresponding terms in the $F(n)$ series and the $f(n)$ series are related. We can use it to transform MHAU's weighted risk indicator, defined in terms of only $F(n)$, into a form involving only $f(n)$. We obtain the relationship:

$$\textbf{Relationship (R1)} \ldots \Sigma F(n) \cdot n = \Sigma f(n) \cdot n \cdot \left[ \frac{(n+1)^{a}}{(n+1)^{a} - (n)^{a}} \right].$$

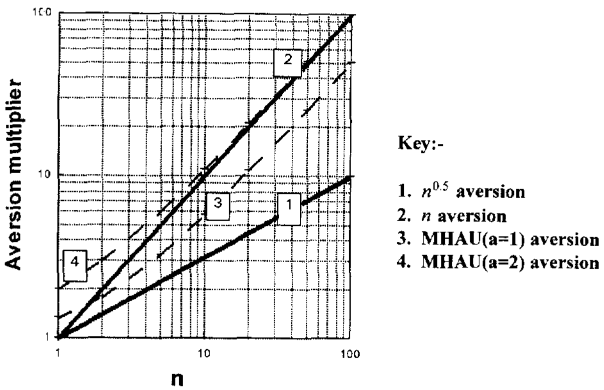


Fig. 1. Comparisons for straight-line $F-n$ curves of unlimited extent.

The right-hand side is now clearly recognisable as an expected disutility function. Moreover, it can be compared easily with the other suggested forms, viz.

$\Sigma f(n) \cdot n^{1.5}$ and $\Sigma f(n) \cdot n^2$

The aversion multiplier applied to each $f(n) \cdot n$ term in the MHAU formulation is different depending on the value of $a$. Recall that $-a$ is the slope of the $F$–$n$ curve. For a slope of $-1$, the multiplier is $(n+1)$, whilst for a slope of $-2$ it is $(n+1)^2/(2n+1)$. Fig. 1 shows how the $n^{0.5}$ and $n$ multipliers and the MHAU $(a=1)$ and $(a=2)$ multipliers vary with $n$. It can be seen that the MHAU formulation is towards the upper end of, but mostly within, the range of the others.

We next consider $F$–$n$ curves which are truncated at some value of $n$, which we will call $N_{max}$. The $F$–$n$ curves of real installations must be truncated, since the number of people at risk from the activities of the installation is surely finite.

Again, we begin with Eq. (1), which remains valid for all $n$. However, for a truncated curve, Eq. (2) is modified as follows:

$$F(n) = \frac{F(1)}{(n)^a} \text{ for } n = 1,2,3,\ldots,N_{max}, \text{ and } F(n) = 0 \text{ for } n > N_{max}. \qquad (2')$$

Following the same logical path which led us to Relationship 1, the derivation now leads to:

$$F(n) = f(n) \times \left[\frac{(n+1)^a}{(n+1)^a - (n)^a}\right] \quad \text{for } n = 1,2,3,\ldots,N_{max}-1,$$

$$F(n) = f(n) \quad \text{for } n = N_{max}, \text{ and}$$

$$F(n) = 0 \quad \text{for } n > N_{max}.$$

Plugging this into MHAU's weighted risk indicator gives:

$$\sum_1^{N_{max}} F(n) \cdot n = \sum_1^{N_{max}-1} f(n) \cdot n \cdot \left[\frac{(n+1)^a}{(n+1)^a - (n)^a}\right] + f(N_{max}) \cdot N_{max}.$$

Making the range of the two summations identical, this is the same as:

$$\sum_{1}^{N_{\max}} F(n)\cdot n = \sum_{1}^{N_{\max}} f(n)\cdot n\cdot\left[\frac{(n+1)^a}{(n+1)^a-(n)^a}\right] - f(N_{\max})\cdot N_{\max}\cdot\left[\frac{(N_{\max}+1)^a}{(N_{\max}+1)^a-(N_{\max})^a}\right] + f(N_{\max})\cdot N_{\max}$$

which simplifies after combining the last two terms to give:

**Relationship (R1′)** ... $$\sum_{1}^{N_{\max}} F(n)\cdot n = \sum_{1}^{N_{\max}} f(n)\cdot n\cdot\left[\frac{(n+1)^a}{(n+1)^a-(n)^a}\right] - f(N_{\max})\cdot N_{\max}\cdot\left[\frac{(N_{\max})^a}{(N_{\max}+1)^a-(N_{\max})^a}\right].$$

Comparing this with Relationship (R1), we see that truncating the $F$–$n$ curve has lost us the neat form of (R1)—an additional term has appeared on the right-hand side. The only real value of Relationship (R1′) is that it demonstrates this effect of truncation. The relationship has no positive value in its own right, the presence of the additional term being seen as an untidy and unwelcome intrusion. Happily, a different relationship, without an intrusive term, can be derived by approaching the task from a different starting point and following a simpler route.

We now begin, not from Eq. (1), but from the fundamental definitions of $f(n)$ and $F(n)$. From them we may write, for our truncated $F$–$n$ curve:

$$\begin{aligned} F(1) &= f(1) + f(2) + f(3) + \ldots + f(N_{\max})\\ F(2) &= \ldots f(2) + f(3) + \ldots + f(N_{\max})\\ F(3) &= \ldots f(3) + \ldots + f(N_{\max}) \\ &\ldots \text{ and so on until} \ldots\\ F(N_{\max}) &= \ldots f(N_{\max}). \end{aligned} \qquad (1'')$$

Plugging these into MHAU's weighted risk indicator gives:

$$\sum_{1}^{N_{\max}} F(n)\cdot n = [f(1)+f(2)+f(3)+\ldots+f(N_{\max})]\times 1 + [f(2)+f(3)+\ldots+f(N_{\max})]\times 2 + [f(3)+\ldots+f(N_{\max})]\times 3 + \ldots + f(N_{\max})\times N_{\max}.$$

Now, because we have a finite number of terms, it is valid to rearrange their order. So, adding up by columns rather than by rows, we can see that

$$\sum_{1}^{N_{\max}} F(n)\cdot n = f(1)\times 1 + f(2)\times(1+2) + f(3)\times(1+2+3) + \ldots + f(N_{\max})\times(1+2+3+\ldots+N_{\max}).$$

But we know that $(1 + 2 + 3 + \ldots + k) = k \cdot (k + 1)/2$. So we can write:

$$\textbf{Relationship (R2)} \ldots \sum_{1}^{N_{max}} F(n) \cdot n = \sum_{1}^{N_{max}} f(n) \cdot n \cdot [n+1]/2.$$

Now we have no intrusive term. Again the right-hand side is recognisable as an expected disutility function, and again we are in a position to make the same comparisons as before. For truncated $F$–$n$ curves MHAU's aversion multiplier is $(n+1)/2$, which again we must compare with $n^{0.5}$ and $n$. Fig. 2 shows the comparison; the broken line being the MHAU formulation. It can be seen that the MHAU formulation lies comfortably within the range of the others for all values of $n$.

Note that in deriving Relationship (R2) we have made no use of Eq. (2). Indeed, we have made no assumption whatsoever about the shape of the $F$–$n$ curve, other than that it is truncated. Relationship (R2) therefore applies to all truncated $F$–$n$ curves.

Incidentally, another useful relationship can be derived from the equations set down above in deriving R2. Adding together the $N_{max}$ rows of Eq. (1″), we obtain:

$$\sum_{1}^{N_{max}} F(n) = f(1) \times 1 + f(2) \times 2 + f(3) \times 3 + \ldots + f(N_{max}) \times N_{max}.$$

That is,

$$\sum_{1}^{N_{max}} F(n) = \sum_{1}^{N_{max}} f(n) \cdot n.$$

Thus, the sum of the summated frequencies gives us the ENFY, from which we began. Again, this relationship applies to all forms of truncated $F$–$n$ curves, irrespective of their shape.

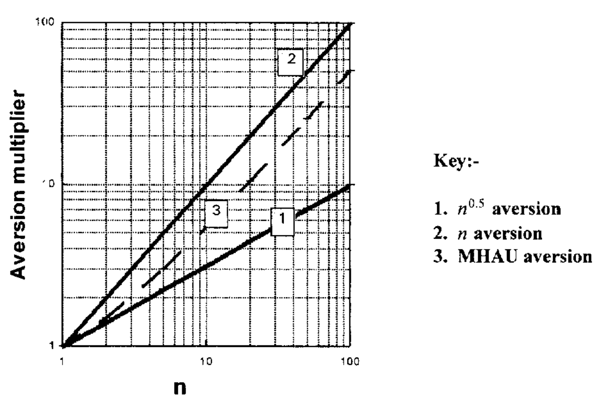


Fig. 2. Comparisons for truncated $F-n$ curves of any shape.

## 3. Summary and conclusions

For $F$–$n$ curves of the form usually presented (straight lines of unlimited extent), the weighted risk indicator used by MHAU in formulating advice to local planning authorities, $\Sigma F(n) \cdot n$, can also be written as:

$$\Sigma f(n) \cdot n \cdot \left[ \frac{(n+1)^a}{(n+1)^a - (n)^a} \right]$$

where $-a$ is the slope of the $F$–$n$ curve. This alternative form makes the indicator readily recognisable as an expected disutility function, suitable (according to classical decision theory) for reaching consistent decisions on risk tolerability. For typical slopes of around $-1$ or $-2$, the term in square brackets, which can be regarded as the aversion multiplier, is towards the upper end of, but mostly within, the range of formulations that have been proposed by others.

For $F$–$n$ curves which are truncated at some maximum value of $n$ (and are therefore, of more practical relevance), MHAU's weighted risk indicator can also be written as:

$$\sum_{1}^{N_{max}} f(n) \cdot n \cdot (n+1)/2.$$

Again, this is recognisable as an expected disutility function, and in this case, the aversion multiplier lies comfortably within the range of formulations proposed by others. The formulation applies to all truncated $F$–$n$ curves, irrespective of their shape.

## 4. Disclaimer

The views expressed in this paper are those of the author and should not be used as a definitive statement of HSE policy.

## References

[1] A.W. Evans, N.Q. Verlander, What is wrong with criterion *FN*-lines for judging the tolerability of risk?, Risk Analysis 17 (2) (1997) 157–168.
[2] D.V. Lindley, Making Decisions, Wiley, 1985.
[3] Health and Safety Executive, Risk Criteria for Land-Use Planning in the Vicinity of Major Industrial Hazards, HMSO, 1989.
[4] D.A. Carter, Health and Safety Executive, Paper published in the Proc. of the 8th Int. Symposium on Loss Prevention and Safety Promotion in the Process Industries, Antwerp, Vol. II, 1995, pp. 219–224.
[5] S.L. Schofield, A framework for offshore risk criteria, Safety and Reliability Soc. J. 13 (2) (1993).
[6] Okrent, Proc. R. Soc. Lond. A 376 (1981) 133–149.